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A MODIFIED SCHWARZSCHILD-TYPE BLACK HOLE SOLUTION AND ITS HORIZON RADIUS IN $f(R)$ GRAVITY

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Abstract

The Schwarzschild solution is fundamental to general relativity (GR) theories, and any alternative theory of modified gravity must replicate it under appropriate conditions. The analysis of Schwarzschild black holes is a significant framework for exploring alternative gravitational theories, such as $f(R)$ gravity, which may arise from constant curvature issues.In this paper we have examined the Schwarzschild black holes solution within the framework of $f(R)$ theories of gravity. In addition, we study under which conditions our new metric coincides with Schwarzschild black holes and SdS black holes in GR. The results have significant implications for the observational differentiation between General Relativity and $f(R)$ gravity theories, particularly in the context of strong gravitational fields. This study also examines the physical impact of black hole horizon structure in the context of $f(R)$ gravity.

Keywords: Schwarzschild black hole, Schwarzschild radius,critical radius, $f(R)$ gravity.

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1 Introduction

Later, many were intrigued by the altered gravity hypotheses and black holes for a long time. A modification of the laws of gravitation on scales where Newtonian gravity or general relativity (GR) has not been thoroughly verified is an alternative answer to the problem of the dynamics of galaxies and galaxy clusters [5]. Without the necessity for dark matter in the current epoch of the universe [2, 3, 4, 5], one such framework known as $f(R)$ gravity has been able to explain

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the dynamics of galaxies and galaxy clusters. The general relativity field content has been augmented to incorporate scalar fields and a massive vector field in one formulation of the $f(R)$ gravity known as the Scalar-Tensor-Vector Gravity (STVG) theory [6].

$f(R)$ gravity is a variation on General Relativity (GR) that broadens Einstein's theory by substituting the Ricci scalar R in the Einstein-Hilbert action with a more general function $f(R)$. This hypothesis falls into the modified gravity theories that seek to explain cosmic acceleration, dark energy, and other gravitational phenomena without relying on exotic forms of matter. Spherically symmetric solutions of f(R)-gravity have recently attracted increasing attention. Solutions in vacuum have been found in [7] by imposing a constant Ricci curvature scalar or considering relations among functions that determine the spherical metric.

$f(R)$ gravity theories have recently gained acceptance as a solid challenge to general relativity, opening fresh perspectives on investigating black hole characteristics. Modified gravity based on the Ricci vector curvature is one theory that changes the Einstein-Hilbert action. The gravitational field equations are altered, which has fascinating repercussions on the characteristics of black holes.

The advantage of utilizing $f(R)$ gravity is that it is built from a fully covariant adaptation of the Einstein-Hilbert action in GR, even if $f(R)$ gravity fits the galaxy rotation curves as well as the phenomenological MOND [8] does. As a result,$f(R)$ gravity can be used when the gravitational field is strong and the weak field approximation fails. The modified Newtonian potential can be obtained from a covariant formalism in many different methods, which should be highlighted.

In this paper, we aim to broaden our understanding of gravitational physics by investigating variations on the classical Schwarzschild black hole in the context of $f(R)$ gravity theories. The Schwarzschild black hole, which is derived from Einstein's field equations in $f(R)$ gravity, defines the gravitational field surrounding a spherically symmetric, non-rotating mass and plays a vital role in understanding black holes. Also, we have derived the radius of our new metric, and then we have to show that under which condition it reduces to the Schwarzschild radius $(r = 2m)$ in GR. The Schwarzschild black hole in $f(R)$ gravity has significant implications for understanding the radius of a black hole's event horizon. The findings of this study contribute to the broader discussion of $f(R)$ gravity theories and their implications for gravitational phenomena.

The paper is arranged as follows. In Sec. 2, we established the Schwarzschild black hole with the help of $f(R)$ gravity action. In Sec. 3, we derived the radius of Schwarzschild black holes in $f(R)$ gravity. The conclusion is presented in Sec. 4.

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2 Solutions in $f(R)$ theories of gravity

For big r, the metric of a star with mass M is

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

It can also be written as

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2}$$

$$\text{where } \; g_{tt} = (1-\frac{2M}{r}) \; \text{ and } \; g_{tt} = (1-\frac{2M}{r})^{-1} \tag{3}$$

In space-time, this is referred to as the Schwarzschild black hole. For very very small values of r, there exists a singularity, which is shown in figure 1(a), 1(b), 1(c). i.e., As $r \to 0$, the term g_{rr} tends to negative infinity, which shows that the curvature of spacetime diverges. For $r = 2M$, the term g_{tt} appears to diverge, which indicates the coordinate singularity.

Let's first examine spherically symmetric exact solutions in $f(R)$-gravity. The relationship between the metric potentials and the Ricci scalar is critical, effectively acting as a constraint equation.

Consider the Ricci scalar R's four-dimensional analytic function $f(R)$. The action of $f(R)$ is varied by

$$S_{VA} = \int d^4x\sqrt{-g}\left[f(R) + \Lambda L_M\right], \tag{4}$$

where

$$\Lambda = \frac{8\pi G}{c^4}, \tag{5}$$

where g is the metric's determinant and L_m is the standard matter lagrangian.

The field equation given in Ref. [9] is obtained by changing the metric, and the metric trace is given by [10]

$$\left.\begin{aligned} F_{\mu\nu} = f^{'}(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} \\ -f^{'}(R)_{\mu\nu} + g_{\mu\nu}\Box f^{'}(R) = \Lambda T_{\mu\nu}. \\ F = 3\Box f^{'}(R) + f^{'}(R)R - 2f(R) = \Lambda T. \end{aligned}\right\} \tag{6}$$

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where

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\ \ L_M)}{\delta g^{\mu\nu}}, \tag{7}$$

is the fluid's standard energy-momentum tensor. The higher than second-order contributions to a filed equation in $f(R)$ gravity [9] can be restated in an Einsteinian form by recasting them as effective stress-energy tensors and modified stress-energy tensors:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T^{eff}_{\mu\nu} + T^{f(R)}_{\mu\nu}.$$

In this case, the effective curvature stress-energy tensor [14]

$$\begin{aligned} T^{eff}_{\mu\nu} = \frac{1}{f'(R)} [g_{\mu\nu}(f(R) - Rf'(R)) \\ + f'(R)^{;\rho\sigma}(g_{\mu\rho}g_{\nu\sigma} - g_{\rho\sigma}g_{\mu\nu})], \end{aligned} \tag{8}$$

and stress-energy tensor that has been modified,

$$T^{f(R)}_{\mu\nu} = \frac{\Lambda T_{\mu\nu}}{f'(R)}. \tag{9}$$

The most general spherically symmetric solution can be written as follows:

$$\begin{aligned} ds^2 = h_1(t', r')dt'^2 + h_2(t', r')dr'^2 + h_3(t', r')dt'dr' \\ + h_4(t', r')d\Omega, \end{aligned} \tag{10}$$

where h_i are the function of r' and t'. We can consider a coordinate transformation such that the off-diagonal terms are vanishes and $h_4(t', r') = r^2$, which maps the metric Eq. (10) that is:

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\Omega. \tag{11}$$

Without losing generality, this formulation can represent the most general definition of a spherically symmetric metric compatible with a pseudo-Riemannian manifold without torsion.
Adding this measure to the fields equations and Eq.(6) really yields the following results:

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} + F_{\mu\nu} = \Lambda T_{\mu\nu}, \tag{12}$$

$$g^{\sigma\tau} F_{\sigma\tau} = f'(R)R - 2f(R) + F = \Lambda T, \tag{13}$$

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where the two quantities $\digamma_{\mu\nu}$ and $\digamma$ are:

$$\begin{aligned}
&\digamma_{\mu\nu} = -f''(R)[R_{,\,\mu\nu} - \Gamma^{t}_{\mu\nu}R_{,\,t} - \Gamma^{r}_{\mu\nu}R_{,\,r}\\
&-g^{\mu\nu}[(g^{tt}_{,\,t} + g^{tt}(log\sqrt{-g})_{,\,t})R_{,\,t}\\
&+(g^{rr}_{,\,r} + g^{rr}(log\sqrt{-g})_{,\,r})R_{,\,r},\\
&+(g^{tt}R_{,\,tt} + g^{rr}R_{,\,rr}]]\\
&-f'''(R)[R_{,\,\mu}R_{,\,\nu} - g_{\mu\nu}(g^{tt}R_{,\,t^2}\\
&+(g^{rr}R_{,\,r^2})],
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\digamma = g^{\sigma\tau}\digamma_{\sigma\tau} =\\
&3f''(R)[(g^{tt}{}_{,\,t} + g^{tt}(log\sqrt{-g})_{,\,t})R_{,\,t}\\
&+(g^{rr}_{,\,r} + g^{rr}(log\sqrt{-g})_{,\,r})R_{,\,r}\\
&+(g^{tt}R_{,\,tt} + g^{rr}R_{,\,rr}]\\
&+3f'''(R)\left[(g^{tt}R_{,\,t^2} + (g^{rr}R_{,\,r^2})\right],
\end{aligned} \tag{15}$$

where the standardized Christoffel symbols for metric $g_{\mu\nu}$ are represented by $\Gamma^{\alpha}_{\mu\nu}$. The intriguing aspect is time and space. This feature will help us better comprehend solutions like dynamical behavior as we shall see shortly.

In writing, Ricci Scalar looks as

$$\begin{aligned}
&R = g^{\mu\nu}R_{\mu\nu} =\\
&g^{\mu\nu}\left[\Gamma^{\alpha}_{\mu\nu\ ,\ \alpha} - \Gamma^{\alpha}_{\mu\alpha\ ,\ \nu} + \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\nu\alpha}\right],
\end{aligned} \tag{16}$$

where $\Gamma^{\alpha}_{\mu\nu}$ is the definition of the Christoffel symbols and defined as [11]

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\rho}(g^{\mu\rho\ ,\ \nu} + g^{\nu\rho\ ,\ \mu} - g^{\mu\nu\ ,\ \rho}). \tag{17}$$

By imposing spherical symmetry, the Ricci scalar is represented in terms of gravitational potentials. Eq. (9) gives:

$$\begin{aligned}
R(t,r) &= \frac{N[\dot{M}\dot{N} - M'^2]r^2}{2r^2M^2N^2}\\
&+\frac{M[r(\dot{N}^2 - \dot{M}\dot{N}) + 2N(2M' + rM'' - r\ddot{N})]}{2r^2M^2N^2}\\
&-\frac{4M^2[N^2 - N + rN']}{2r^2M^2N^2}.
\end{aligned} \tag{18}$$

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The explicit dependence in $M(t, r)$ and $N(t, r)$ has been eliminated for conciseness, and the prime represents the derivative with respect to r. In contrast, the dot represents the derivative concerning t. If the metric Eq. (11) is independent of time, i.e., $M(r, t) = m(r), N(r, t) = n(r)$ Eq. (18) assuming the more basic form:

$$R(r) = \frac{m(r)[2n(r)(2m'(r) + rm''(r)) - rm'(r)n'(r)]}{2r^2m(r)^2n(r)^2} \\ - \frac{m(r)[n(r)m'(r)^2r^2 - 4m(r)^2(n(r)^2 - n(r) + rn'(r))]}{2r^2m(r)^2n(r)^2}, \tag{19}$$

where the explicit demonstration of the gravitational potential's radial dependency is made. This formula is a constraint for the functions $m(r)$ once a specified form of the Ricci scalar is provided, $n(r)$. Particularly, it is reduced to a Bernoulli equation of index two, which is

$$n'(r) + l(r)n(r) + h(r)n(r)^2 = 0.$$

Regarding the metric potential $n(r)$:

$$n'(r) + \\ \left[\frac{r^2m'(r)^2 - 4m(r)^2 - 2rm(r)[2m'(r) + rm''(r)]}{ra(r)[4m(r) + rm'(r)]}\right]n(r) \\ + \left[\frac{2m(r)}{r}\left[\frac{2 + r^2R(r)}{4m(r) + rm'(r)}\right]\right]n(r)^2 = 0. \tag{20}$$

After some mathematics, we get the general solution of Eq. (20):

$$n(r) = \frac{[-\int l(r)dr]}{\zeta + \int h(r)drexp[-\int l(r)dr]}, \tag{21}$$

Where ζ is an integration constant and $l(r)$ and $h(r)$ are the two functions used in the ordinary Bernoulli equation to specify the coefficients of the quadratic and linear terms concerning $n(r)$ [12]. By looking at the equation, we can observe that $h(r) = 0$, meaning that we can find solutions with a Ricci scalar scaling of $\frac{-2}{r^2}$ in terms of the radial coordinate. On the other hand, $l(r)$ cannot be equal to zero because this would result in illogical solutions. The limit r is appropriate for specific consideration. Given that $r^2R(r)$ is a constant quantity, $n'(r) = 0$ means that both $l(r)$ and $h(r)$ must be zero to produce a gravitational potential with the correct Minkowski limit. The potential metric $n(r)$ has the correct Minkowski limit.

In general, the Ricci scalar must evolve at infinity as r^{-n} with $n \geq 2$ if we require the

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asymptotic flatness of the metric as a property of the theory. Formally, it must be:

$$\lim_{r\to\infty} r^2 R(r) = r^{-n}, \tag{22}$$

with $n \in N$. The criterion for the right asymptotic flatness could be affected by any other behavior of the Ricci scalar. This conclusion can be drawn from Eq. (20). Let's think about the most straightforward spherically symmetric situation:

$$ds^2 = -m(r)dt^2 + \frac{dr^2}{m(r)} + r^2 d\Omega. \tag{23}$$

The Bernoulli Eq. (20) is easy to integrate, and the most general metric potential m(r), compatible with the Ricci scalar constraint Eq. (19), is :

$$m(r) = 1 + \frac{\zeta_1}{r} + \frac{\zeta_2}{r} + \frac{1}{r^2}\int\Big[\int r^2 R(r)dr\Big]dr, \tag{24}$$

where ζ_1 and ζ_2 are integration constants. One gets the expected result $m(r) = 1$ (Minkowski) for $r \to \infty$ only if the condition Eq. (22) is satisfied. Otherwise, we get a diverging gravitational potential.

The solution is time-independent and identical to GR with a cosmological constant for constant curvature. This conclusion is well known (see, for instance, [13]), but we describe it for completeness to deal with a more widespread case where a radial dependence is assumed. Let us assume a scalar curvature constant($R = R_0$). The field Eq. (12) and Eq. (13) being $F_{\mu\nu} = 0$ reduce to:

$$f_0' R_{\mu\nu} - \frac{1}{2} f_0 g_{\mu\nu} = \Lambda T_{\mu\nu}. \tag{25}$$

$$f_0' R_0 - 2f_0 = \Lambda T. \tag{26}$$

Where $f(R_0) = f_0, f'(R_0) = f_0'$. Such an equation can be regarded as

$$R_{\mu\nu} - \lambda f_0 g_{\mu\nu} = q\Lambda T_{\mu\nu}. \tag{27}$$

$$R_0 = q\Lambda T - 4\lambda. \tag{28}$$

Where $\lambda = \frac{-f_0}{2f_0'}$ and $q^-1 = f_0'$. Typically, the perfect-fluid matter's stress-energy tensor is

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta}. \tag{29}$$

Where the energy density is ρ , the pressure is p and $u^\alpha = \frac{dx^\alpha}{ds}$ is the 4-velocity. The above set

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of equations receives a general solution for $p = -\rho$ and reads :

$$ds^2 = -(1 - \frac{\zeta_1}{r} + \frac{q\Lambda\rho - 2\lambda}{3}r^2)dt^2 + \frac{dr^2}{(1 - \frac{\zeta_1}{r} + \frac{q\Lambda\rho-2\lambda}{3}r^2)} + r^2 d\Omega. \tag{30}$$

Where $\zeta_1 = \frac{2GM}{c^2}$. This finding implies that every $f(R)$ theory exhibits solutions with cosmological constant Schwarzschild de Sitter-like behavior for the situation of constant curvature scalar $(R = R_0)$. Eq. (30) represents the necessary Schwarzschild de Sitter black hole for $f(R)$ gravity. The metric (30) has two singularities: (i) a coordinate singularity (horizontal) where $g_{rr} \to \infty$, and (ii) a weakened true physical singularity at $r \to 0$ In figures 2(a), 2(b), and 2(c), we have plotted the singularities for different masses in $f(R)$ theories of gravity.

Eq(30) can also be written as

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\Omega. \tag{31}$$

$$\text{where } g_{tt} = (1 - \frac{\zeta_1}{r} + \frac{q\Lambda\rho - 2\lambda}{3}r^2) \tag{32}$$

$$\text{and } g_{rr} = \frac{1}{(1 - \frac{\zeta_1}{r} + \frac{q\Lambda\rho-2\lambda}{3}r^2)} \tag{33}$$

Let $l^2 = -\frac{3}{q\Lambda\rho-2\lambda}$, metric (30) reduces to the Schwarzschild de Sitter metric. It can be written thus,

$$ds^2 = -(1 - \frac{\zeta_1}{r} - \frac{r^2}{l^2})dt^2 + \frac{dr^2}{(1 - \frac{\zeta_1}{r} - \frac{r^2}{l^2})} + r^2 d\Omega, \tag{34}$$

where $\zeta_1 = 2M, G = c = 1$. Hence, we can conclude that in $f(R)$ gravity, Schwarzschild black hole act as Shcwarzschild de-Sitter black hole.

Case: If we set $\Lambda = \lambda = 0$ Eq. (30) reduces to the metric (1) in general relativity.

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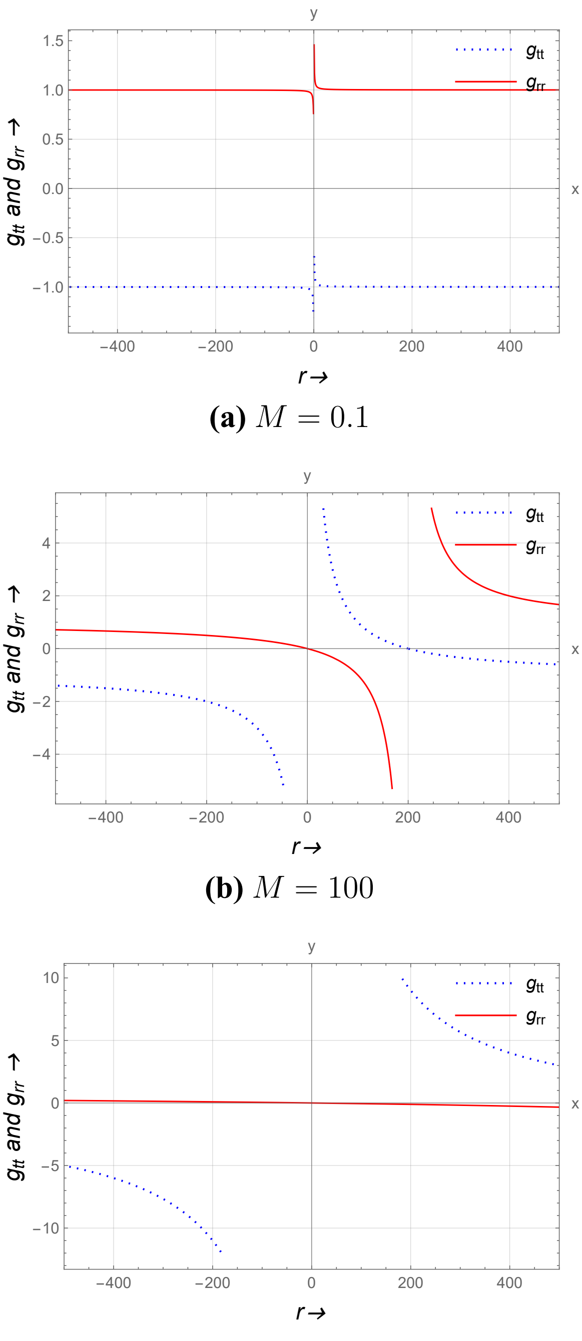
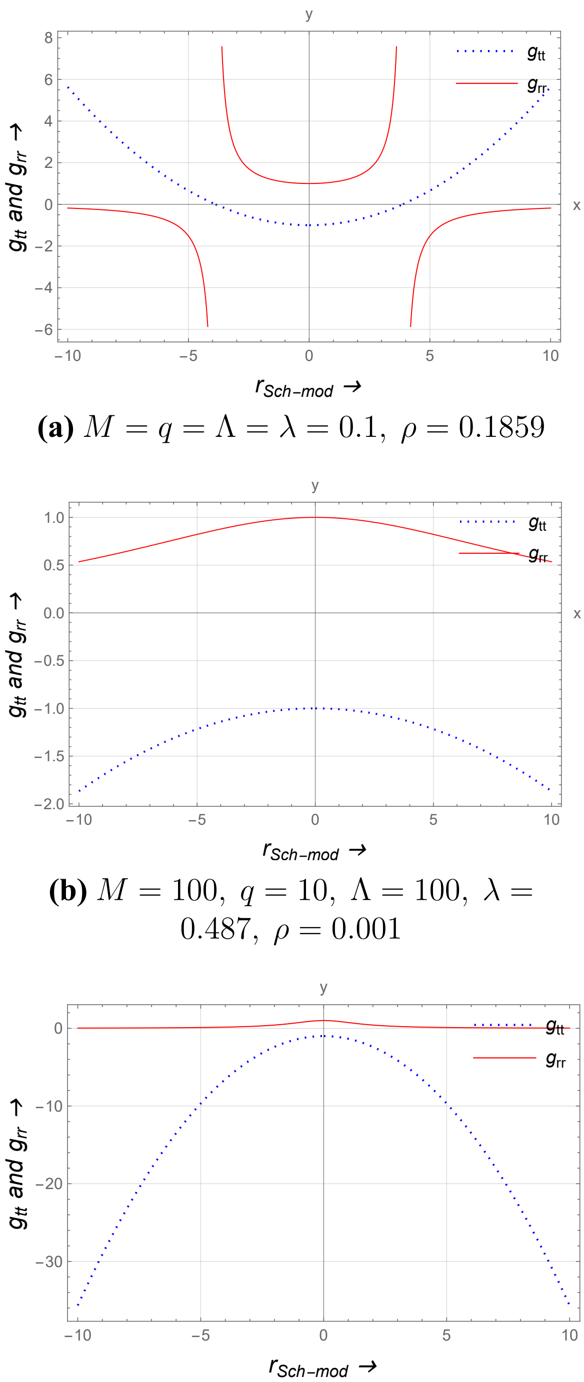


(a) $M = 0.1$

(b) $M = 100$

(c) $M = 1000$

Figure 1: Graph of equation (1) in GR for different masses.



(a) $M = q = \Lambda = \lambda = 0.1,\ \rho = 0.1859$

(b) $M = 100,\ q = 10,\ \Lambda = 100,\ \lambda = 0.487,\ \rho = 0.001$

(c) $M = 1000,\ q = 10,\ \Lambda = 3,\ \lambda = 0.1,\ \rho = 0.01$

Figure 2: Graph of equation (30) in $f(R)$ gravity.

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3 Schwarzschild radius in $f(R)$ gravity

The gravitational or Schwarzschild radius is the distance below which irreversible gravitational collapse must result from the gravitational attraction between the constituent particles of the body. The more massive stars are believed to have this as their final destiny. In the formula below, where G denotes the gravitational constant of the universe, and c represents the light speed, the Schwarzschild radius (r_{Sch}) of an object of mass M is denoted:

$$r_{Sch} = \frac{2GM}{c^2}. \tag{35}$$

We find the Schwarzschild radius in $f(R)$ gravity. The Schwarzschild black hole in $f(R)$ gravity is as follows:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega. \tag{36}$$

$$\text{where} f(r) = \big(1 - \frac{\zeta_1}{r} + \frac{q\Lambda\rho - 2\lambda}{3} r^2\big). \tag{37}$$

Singularity exist at $f(r) = 0$, which gives the position of horizon as

$$r = \zeta_1 (1 - \frac{q\Lambda\rho - 2\lambda}{3} r^2). \tag{38}$$

With the initial conditions $r = \zeta_1 = \frac{2GM}{c^2}$ we can developed the Schwarzschild radius in $f(R)$ gravity as follows

$$r_{Sch-f(R)} = \frac{2GM}{c^2}\left(1 - \frac{q\Lambda\rho - 2\lambda}{3}\, \frac{4G^2M^2}{c^4}\right). \tag{39}$$

Now, we have developed the critical radius of a star in the case of the weak field approximation. For this, we have

$$g_{00} = \big(1 - \frac{\zeta_1}{r} + \frac{q\Lambda\rho - 2\lambda}{3} r^2\big), \tag{40}$$

minimizing the energy E due to the weak field approximation, the critical radius may be found using the following energy for the generalized special relativity equilibrium condition [14]:

$$E = \frac{m_0 c^2 g_{00}}{\sqrt{g_{00} - \frac{v^2}{c^2}}}. \tag{41}$$

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The conditions for minimum value, i.e., can be used to determine the critical radius of the star, which necessitates minimizing the total energy E, i.e.,

$$\frac{dE}{dr} = 0. \tag{42}$$

After some mathematics, we get the critical radius corresponding to the modified radius of the Schwarzschild black hole as

$$r_{c-f(R)} = r_{Sch-f(R)} = \frac{2GM}{c^2}\left(1 - \frac{q\Lambda\rho - 2\lambda}{3}\frac{4G^2M^2}{c^4}\right). \tag{43}$$

M was previously a star's mass before disintegrating into a black hole. Schwarzschild radius' equivalent radius is r. As a result, we discovered the following formula for a black hole's Schwarzschild radius in $f(R)$ gravity:

$$r_{Sch-f(R)} = \frac{2GM}{c^2}\left(1 - \frac{q\Lambda\rho - 2\lambda}{3}\frac{4G^2M^2}{c^4}\right). \tag{44}$$

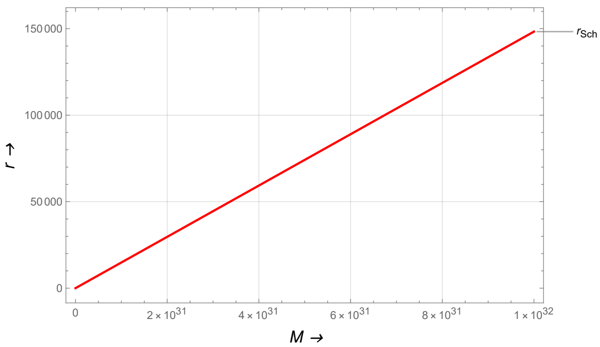

Figure 3: Radius vs Mass in GR. Here, we can see that the radius and mass of the metric (1) are proportional in GR.

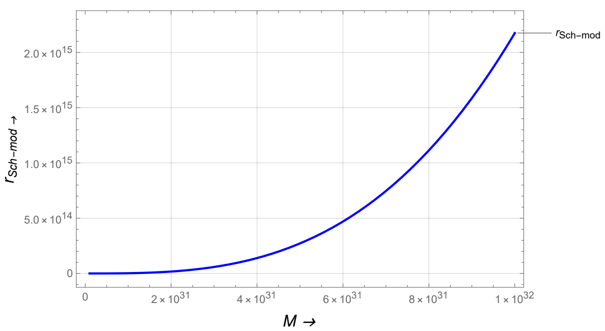

Figure 4: Radius vs Mass in $f(R)$ gravity (q =1, λ =1, $\rho = 1$). In this case, we observe that the metric radius found (30) increases slowly as mass increases.

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4 CONCLUSION

This paper presents a modified Schwarzschild solution using $f(R)$ gravity. Our approach successfully recovers the Schwarzschild-de Sitter metric for $l^2 = -\frac{3}{q\Lambda\rho-2\lambda}$, thanks to careful parameter selection. In the case when $\Lambda = \lambda = 0$, our solution precisely reduces to the conventional Schwarzschild metric in general relativity (GR), demonstrating the model's compatibility with existing results in the proper conditions. We compared the radius of the black hole horizon in GR and $f(R)$ gravity across different black hole masses to see how updated gravity terms affect it. Figures 3 and 4 demonstrate measurable variances, demonstrating the scientific significance of $f(R)$ corrections in strong gravitational regimes. The present work could investigate the thermodynamic and quantum features of the resulting black hole solution, such as Hawking radiation and entropy corrections. Extending the analysis to charged or rotating black holes and comparing with observable data can help limit viable $f(R)$ gravity models.

References

[1] J. W. Moffat, "Scalar-Tensor-Vector Gravity," JCAP **03**, 004 (2006).

[2] J. W. Moffat and S. Rahvar, "The MOG weak field approximation and observational test of galaxy rotation curves," MNRAS **436**, 1439 (2013).

[3] J. W. Moffat and S. Rahvar, "The MOG weak field approximation - II. Observational test of Chandra X-ray clusters," MNRAS **441**, 3724 (2014).

[4] J. R. Brownstein and J. W. Moffat, "Galaxy cluster masses without non-baryonic dark matter," MNRAS **367**, 527 (2006).

[5] J. W. Moffat and V. T. Toth, "Rotational velocity curves in the Milky Way as a test of modified gravity," Phys. Rev. D **91**, 043004 (2015).

[6] J. W. Moffat, "Structure Growth and the CMB in Modified Gravity (MOG)," arXiv:1409.0853 [astro-ph.CO] (2014).

[7] T. Multamaki and I. Vilja, "Spherically symmetric solutions of modified field equations in $f(R)$ theories of gravity," Phys. Rev. D **74**, 064022 (2006).

[8] M. Milgrom, "A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis," Astrophys. J. **270**, 365 (1983).

[9] Pnaki Q. Roy, "Field equations in $f(R)$ gravity," Christ University (2019).

[10] S. Capozziello, V. F. Cardone, and A. Troisi, "Dark energy and dark matter as curvature effects," JCAP **08**, 001 (2006).

Online ISSN: 3048-667X
CODEN (CAS, USA): JRLAB4
Impact Factor: 3.66

[11] T. P. Sotiriou and V. Faraoni, "$f(R)$ theories of gravity," Rev. Mod. Phys. **82**, 451 (2010).

[12] Rafael Ferraro, "$f(R)$ and $f(T)$ theories of modified gravity",doi.org/10.1063/1.4756821(2012).

[13] T. P. Sotiriou, "$f(R)$ gravity and scalar-tensor theory", Class. Quant. Grav. **23**, 5117. (2006).

[14] Â¹Mohammed Saeed Dawood Aamir, 'Mubarak Dirar Abdallah, and'Sawsan Ahmed Elhouri Ahmed. "Derivation of the Schwarzschild Radius Formula on the Basis Generalized Special Relativity",(IJSEAS)-Volume-8, Issue-8, August 2022